ARL-TR-8709 ● June 2019

Invariant Vector Representation of a Qutrit in a Cascade Model

by Vinod K Mishra

NOTICES

Disclaimers

ARL-TR-8709 ● JUNE 2019

Invariant Vector Representation of a Qutrit in a Cascade Model

by Vinod K Mishra
Computational and Information Sciences Directorate, CCDC Army Research Laboratory

REPORT DOCUMENTATION PAGE						*Form Approved* *OMB No. 0704-0188*
Public reporting burden for this collection of information is estimated to average 1 hour per response, including the time for reviewing instructions, searching existing data sources, gathering and maintaining the data needed, and completing and reviewing the collection information. Send comments regarding this burden estimate or any other aspect of this collection of information, including suggestions for reducing the burden, to Department of Defense, Washington Headquarters Services, Directorate for Information Operations and Reports (0704-0188), 1215 Jefferson Davis Highway, Suite 1204, Arlington, VA 22202-4302. Respondents should be aware that notwithstanding any other provision of law, no person shall be subject to any penalty for failing to comply with a collection of information if it does not display a currently valid OMB control number. **PLEASE DO NOT RETURN YOUR FORM TO THE ABOVE ADDRESS.**						
1. REPORT DATE *(DD-MM-YYYY)* June 2019		**2. REPORT TYPE** Technical Report				**3. DATES COVERED (From - To)** 1 February–31 March 2019
4. TITLE AND SUBTITLE Invariant Vector Representation of a Qutrit in a Cascade Model						**5a. CONTRACT NUMBER**
						5b. GRANT NUMBER
						5c. PROGRAM ELEMENT NUMBER
6. AUTHOR(S) Vinod K Mishra						**5d. PROJECT NUMBER**
						5e. TASK NUMBER
						5f. WORK UNIT NUMBER
7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES) US Army Combat Capabilities Development Command Army Research Laboratory ATTN: FCDD-RLC-NC Aberdeen Proving Ground, MD 21005						**8. PERFORMING ORGANIZATION REPORT NUMBER** ARL-TR-8709
9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)						**10. SPONSOR/MONITOR'S ACRONYM(S)**
						11. SPONSOR/MONITOR'S REPORT NUMBER(S)
12. DISTRIBUTION/AVAILABILITY STATEMENT Approved for public release; distribution is unlimited.						
13. SUPPLEMENTARY NOTES ORCID ID: Vinod K Mishra, 0000-0001-9432-9082						
14. ABSTRACT The qutrit state density matrix is of order 3 and depends on eight parameters in the most general case. Visualization of this 8-D state space is practically impossible with commonly used 8-D vectors. Recently, a 3-D vector representation of the qutrit state space (also called invariant vector representation [IVR]) was proposed. In this report, we present the time–evolution of the IVR vectors for the qutrit cascade or Ξ-model to emphasize the advantages of the IVR.						
15. SUBJECT TERMS quantum density of states, qubits, qutrits, bloch sphere, invariant vector representation						
16. SECURITY CLASSIFICATION OF:				**17. LIMITATION OF ABSTRACT**	**18. NUMBER OF PAGES**	**19a. NAME OF RESPONSIBLE PERSON** Vinod K Mishra
a. REPORT Unclassified	**b. ABSTRACT** Unclassified	**c. THIS PAGE** Unclassified		UU	16	**19b. TELEPHONE NUMBER (Include area code)** 410-278-0114

Standard Form 298 (Rev. 8/98)
Prescribed by ANSI Std. Z39.18

Contents

List of Figures

1. Introduction

The qutrit (three internal states) comes next in complexity after qubit (two internal states) as a resource for quantum information processes like quantum sensing.[1] The qubit density matrix is of order 2 and it depends on three parameters for the most general mixed states, but only on two parameters for pure states. It can be easily visualized using Bloch sphere representation in which the pure states are represented by points on the Bloch sphere. On the other hand, the qutrit density matrix is of order 3 and it depends on eight parameters in the most general case. Visualization of the 8-D qutrit state space is practically impossible using the 8-D vectors commonly used in the Gell-Mann representation.[2] Recently, another 3-D representation of the qutrit state space was proposed based on density matrix invariants (called invariant vector representation [IVR]).[3] These vectors also reside on the surface of a sphere and help one visualize the dynamics of a qutrit state. In this report, we apply IVR to the cascade configuration of a qutrit (also known as the $\Xi-$model[2,4]) and show its utility in understanding the dynamics of the model.

2. Invariant Vector Representation (IVR) of a Qutrit: Basic Results

The density matrix ρ based on the spin-1 representation of a qutrit is given as[5,6]

$$\rho = \begin{bmatrix} \omega_1 & \frac{1}{2}(q_3 + ia_3) & \frac{1}{2}(q_2 - ia_2) \\ \frac{1}{2}(q_3 - ia_3) & \omega_2 & -\frac{1}{2}(q_1 + ia_1) \\ \frac{1}{2}(q_2 + ia_2) & -\frac{1}{2}(q_1 - ia_1) & \omega_3 \end{bmatrix}. \tag{1}$$

The parameters of ρ are related to the expectation values of expressions involving spin-1 components and their combinations.

$$\omega_i =< S_i^2 >= Tr(\rho S_i^2)\ , \tag{2a}$$

$$a_i =< S_i >= Tr(\rho S_i)\ , \tag{2b}$$

$$q_k =< S_i S_j + S_j S_i >= Tr\{\rho(S_i S_j + S_j S_i)\}, k \neq i,j\ . \tag{2c}$$

The invariant 3-D vectors are given by the following relations.[5]

First invariant vector:

$$\vec{w} = \{\sqrt{\omega_1},\quad \sqrt{\omega_2},\quad \sqrt{\omega_3}\}\ . \tag{3a}$$

Based on the trace relation,

$$Tr(\rho_{\Xi}) = \omega_1 + \omega_2 + \omega_3 = 1 \, , \sum_{i=1}^{3} w_i^2 = 1 \, . \tag{3b}$$

Second invariant vector:

$$\vec{u} = \left\{\sqrt{\omega_1^2 + (q_1^2 + a_1^2)/2}, \quad \sqrt{\omega_2^2 + (q_2^2 + a_2^2)/2}, \quad \sqrt{\omega_3^2 + (q_3^2 + a_3^2)/2}\right\} . \tag{4a}$$

Based on the trace relation,

$$Tr(\rho_{\Xi}^2) = \sum_{i=1}^{3} u_i^2 \leq 1 \, . \tag{4b}$$

Third invariant vector:

$$\vec{v} = \left\{\sqrt{X + 3(q_1^2 + a_1^2)/2}, \sqrt{X + 3(q_2^2 + a_2^2)/2}, \sqrt{X + 3(q_3^2 + a_3^2)/2}\right\} . \tag{5a}$$

Here,

$$X = \frac{1}{3} - 2\omega_1\omega_2\omega_3 - \frac{1}{2}(a_2a_3q_1 + a_3a_1q_2 + a_1a_2q_3 - q_1q_2q_3) \, . \tag{5b}$$

Based on the trace relation,

$$3Tr(\rho_{\Xi}^2) - 2Tr(\rho_{\Xi}^3) = \sum_{i=1}^{3} v_i^2 \leq 1 \, . \tag{5c}$$

The vectors $\vec{u}$ and $\vec{v}$ represents the second and third density matrix invariants of a qutrit. The bounds on the vector-norms are, in general, $\sum_{i=1}^{3} u_i^2 \leq 1$ and $\sum_{i=1}^{3} v_i^2 \leq 1$, with equality signs holding for a pure state.

For a pure state, the nine density matrix parameters are related via the following five relations.

1 relation

$$Tr(\rho_{\Xi}) = \omega_1 + \omega_2 + 3 = 1 \, , \tag{6a}$$

3 relations

$$\frac{1}{4}\begin{pmatrix} q_1^2 + a_1^2 \\ q_2^2 + a_2^2 \\ q_3^2 + a_3^2 \end{pmatrix} = \begin{pmatrix} \omega_2\omega_3 \\ \omega_3\omega_1 \\ \omega_1\omega_2 \end{pmatrix} , \tag{6b}$$

1 relation

$$a_2a_3q_1 + a_3a_1q_2 + a_1a_2q_3 - q_1q_2q_3 = 8\omega_1\omega_2\omega_3 \, . \tag{6c}$$

So, finally, we have only four independent parameters, which is the correct number of independent degrees of freedom for a pure qutrit state.

3. Time-Dependent Eigenvectors of the Qutrit Cascade or Ξ-Model

The isolated qutrit states with their energies are taken to be $|0>(E_0)$, $|1>(E_1)$, and $|2>(E_2)$ with $E_2 > E_1 > E_0$ average energy $\bar{E} = (E_0 + E_1 + E_2)/3$. Define $\varepsilon_1 = (-2E_0 + E_1 + E_2)/3$, and $\varepsilon_2 = (-E_0 - E_1 + 2E_2)/3$, then the starting energies are related with them as

$$E_0 = \bar{E} - \varepsilon_1 \, , \tag{7a}$$

$$E_1 = \bar{E} - (\varepsilon_2 - \varepsilon_1) = \bar{E} - \varepsilon, \tag{7b}$$

$$E_2 = \bar{E} + \varepsilon_2 \, . \tag{7c}$$

Here $\varepsilon = \varepsilon_2 - \varepsilon_1 = (E_0 + E_2 - 2E_1)/3 > 0$ has been assumed for later analysis ($\varepsilon < 0$ case is similar). After taking the zero of energy at $\bar{E}$, the Hamiltonian of the qutrit in an external field is given by

$$H = \begin{bmatrix} -\varepsilon_1 & (g_1 - ig_2)\phi & (g_3 - ig_4)\phi \\ (g_1 + ig_2)\phi & -\varepsilon & (g_5 - ig_6)\phi \\ (g_3 + ig_4)\phi & (g_5 + ig_6)\phi & \varepsilon_2 \end{bmatrix}. \tag{8}$$

We specialize to the equidistant level Ξ-model in which, (i) $\varepsilon_2 = \varepsilon_1$, (ii) couplings are equal, in other words, $g_1 = g_3 = g_5, g_2 = g_4 = g_6$, and (iii) there is no coupling between first and third levels. Then the Hamiltonian becomes

$$H_\Xi = \begin{bmatrix} -\varepsilon_1 & \phi(g_1 - ig_2) & 0 \\ \phi(g_1 + ig_2) & 0 & \phi(g_1 - ig_2) \\ 0 & \phi(g_1 + ig_2) & \varepsilon_1 \end{bmatrix}. \tag{9}$$

Let

$$g_1 + ig_2 = Ge^{i\delta} \, , \tag{10a}$$

$$\omega = \sqrt{\varepsilon_1^2 + 2\phi^2 G^2}, \tag{10b}$$

$$\varepsilon_1 = \omega cos\theta \, , \tag{10c}$$

$$\phi G\sqrt{2} = \omega sin\theta \, . \tag{10d}$$

Then we get

$$H_{\Xi} = \omega \begin{bmatrix} -cos\theta & \frac{1}{\sqrt{2}} sin\theta e^{-i\delta} & 0 \\ \frac{1}{\sqrt{2}} sin\theta e^{i\delta} & 0 & \frac{1}{\sqrt{2}} sin\theta e^{-i\delta} \\ 0 & \frac{1}{\sqrt{2}} sin\theta e^{i\delta} & cos\theta \end{bmatrix}. \tag{11}$$

The stationary eigenvectors for eigenvalues $(-\omega, 0, \omega)$ of H_{Ξ} are found respectively as

$$|0>= \begin{pmatrix} e^{-i\delta} cos^2 \frac{\theta}{2} \\ -\frac{1}{\sqrt{2}} sin\theta \\ e^{i\delta} sin^2 \frac{\theta}{2} \end{pmatrix}, |1>= \begin{pmatrix} \frac{1}{\sqrt{2}} e^{-i\delta} sin\theta \\ cos\theta \\ -\frac{1}{\sqrt{2}} e^{i\delta} sin\theta \end{pmatrix}, |2>= \begin{pmatrix} e^{-i\delta} sin^2 \frac{\theta}{2} \\ \frac{1}{\sqrt{2}} sin\theta \\ e^{i\delta} cos^2 \frac{\theta}{2} \end{pmatrix}. \tag{12}$$

The time-dependent Schrödinger's equation for the Ξ-model is given as

$$i \frac{\partial}{\partial t} \begin{pmatrix} |\Psi_0(t)> \\ |\Psi_1(t)> \\ |\Psi_2(t)> \end{pmatrix} = H_{\Xi} \begin{pmatrix} |\Psi_0(t)> \\ |\Psi_1(t)> \\ |\Psi_2(t)> \end{pmatrix}. \tag{13}$$

As before, the interacting field ϕ is time-independent and the initial conditions are: (i) $|\psi_0(t=0)>$ = 1, and (ii) $|\psi_1(t=0)>$ = 0 = $|\psi_2(t=0)>$. The solutions for the time-dependent eigenvectors are

$$\begin{pmatrix} |\Psi_0(t)> \\ |\Psi_1(t)> \\ |\Psi_2(t)> \end{pmatrix} = cos^2 \frac{\theta}{2} e^{i(\omega t+\delta)} |0> + \frac{1}{\sqrt{2}} sin\theta e^{i\delta} |1> + sin^2 \frac{\theta}{2} e^{-i(\omega t-\delta)} |2>. \tag{14}$$

They can be rewritten as

$$\begin{pmatrix} |\Psi_0(t)> \\ |\Psi_1(t)> \\ |\Psi_2(t)> \end{pmatrix} = \begin{pmatrix} \left(1 - \frac{1}{2} sin^2\theta\right) cos\omega t + \frac{1}{2} sin^2\theta + i sin\omega t cos\theta \\ \frac{1}{\sqrt{2}} sin\theta [cos\theta(1 - cos\omega t) - i sin\omega t] e^{i\delta} \\ -\frac{1}{2} sin^2\theta (1 - cos\omega t) e^{2i\delta} \end{pmatrix}. \tag{15}$$

4. IVR of a Qutrit: Ξ-Model

The density matrix of the qutrit cascade or Ξ-model is calculated as

$$\rho_{\Xi} = \begin{pmatrix} |\Psi_0(t)> \\ |\Psi_1(t)> \\ |\Psi_2(t)> \end{pmatrix} (<\Psi_0(t)| \quad <\Psi_1(t)| \quad <\Psi_2(t)|). \tag{16}$$

We express the resulting density matrix as the spin-1 representation given earlier. Then for the qutrit Ξ-model, the following are the expressions for the density matrix parameters.

$$\omega_1 = \frac{1}{4}(3 + cos2\theta)cos^2\omega t \ , \tag{17a}$$

$$\omega_2 = \frac{1}{4}(3 + cos2\theta)sin^2\omega t \ , \tag{17b}$$

$$\omega_3 = \frac{1}{4}(1 - cos2\theta) \ , \tag{17c}$$

$$\begin{pmatrix} q_1 \\ a_1 \end{pmatrix} = \frac{sin\theta}{2\sqrt{2}}(1 - cos2\theta)(1 - cos\omega t)\left[cos\theta(1 - cos\omega t)\begin{pmatrix} cos\delta \\ -sin\delta \end{pmatrix} - sin\omega t \begin{pmatrix} sin\delta \\ cos\delta \end{pmatrix}\right], \tag{17d}$$

$$\begin{pmatrix} q_2 \\ a_2 \end{pmatrix} = \frac{1-cos2\theta}{8}(1 - cos\omega t)\left[-\{(1 - cos2\theta) + (3 + cos2\theta)cos\omega t\}\begin{pmatrix} cos2\delta \\ sin2\delta \end{pmatrix} + 4cos\theta sin\omega t\begin{pmatrix} -sin2\delta \\ cos2\delta \end{pmatrix}\right], \tag{17e}$$

$$\begin{pmatrix} q_3 \\ a_3 \end{pmatrix} = \frac{sin\theta}{4\sqrt{2}}cos\theta[-(5 + 3cos2\theta) + 4(1 + cos2\theta)cos\omega t + (1 - cos2\theta)cos2\omega t]\begin{pmatrix} cos\delta \\ -sin\delta \end{pmatrix} + [2(3 + cos2\theta)sin\omega t + (1 - cos2\theta)sin2\omega t]\begin{pmatrix} sin\delta \\ cos\delta \end{pmatrix}. \tag{17f}$$

Due to the structure of the Ξ-model, there are only two independent parameters (ω, θ). Using the earlier expressions, the IVR vectors for Ξ-model are found to be

1) First-order invariant vector ($\vec{w} = \{\sqrt{\omega_1}, \quad \sqrt{\omega_2}, \quad \sqrt{\omega_3}\}$):

The angles in spherical representation are:

Colatitude angle or the angle between the IVR vector and z-axis

$$\psi_1 = cos^{-1}(\sqrt{\omega_3}) = cos^{-1}\left(\frac{1}{\sqrt{2}}sin\theta\right). \tag{18a}$$

It is time-independent.

Azimuthal angle or the angle between projection of IVR on XY-plane and the x-axis

$$\chi_1 = tan^{-1}\sqrt{\frac{\omega_2}{\omega_1}} = \omega t \text{ (modulo } 2\pi) . \tag{18b}$$

It has linear time-dependence.

2) Second-order invariant vector ($\vec{u} = \{\sqrt{\omega_1^2 + 2\omega_2\omega_3}, \quad \sqrt{\omega_2^2 + 2\omega_3\omega_1}, \quad \sqrt{\omega_3^2 + 2\omega_1\omega_2}\}$):

The IVR angles for $\vec{u}(\psi_2, \chi_2)$ are:

Colatitude angle or the angle between the IVR vector and Z-axis:

$$\psi_2 = cos^{-1}\sqrt{\omega_3^2 + 2\omega_1\omega_2} \ , \qquad (19a)$$

Azimuthal angle or the angle between projection of IVR on XY-plane and the X-axis

$$\chi_2 = tan^{-1}\sqrt{\frac{\omega_2^2+2\omega_3\omega_1}{\omega_1^2+2\omega_2\omega_3}} \, . \qquad (19b)$$

We calculate and plot $(\psi_2, \omega t)$ and $(\chi_2, \omega t)$ for values of $\theta = 3.0$ radians and 5.0 radians as shown in Figs. 1–4.

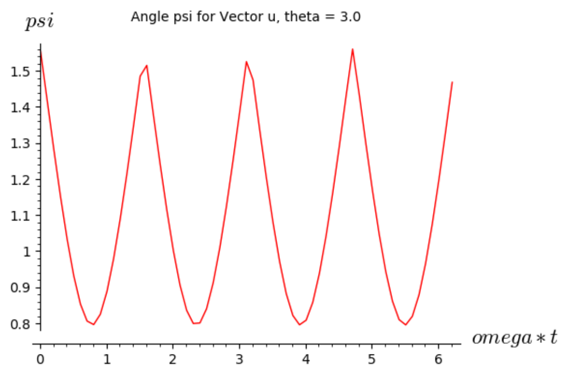

Fig. 1 Colatitude angle of the second-order invariant vector u as a function of ωt for θ = 3 radians

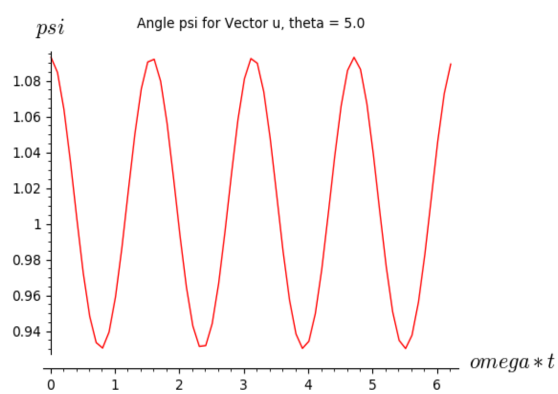

Fig. 2 Colatitude angle of the second-order invariant vector u as a function of ωt for θ = 5 radians

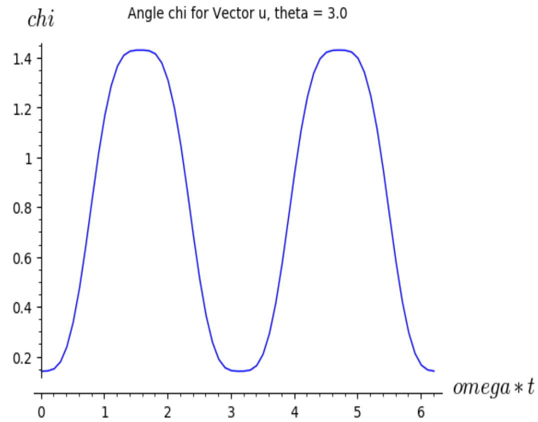

Fig. 3 Azimuthal angle of the second-order invariant vector u as a function of ωt for θ = 3 radians

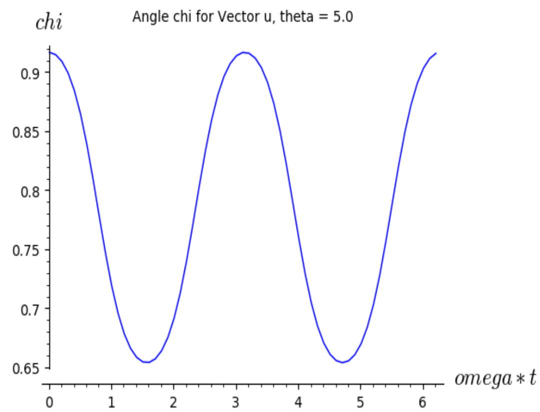

Fig. 4 Azimuthal angle of the second-order invariant vector u as a function of ωt for θ = 5 radians

These angles show sinusoidal-like variations of the spherical angles of IVR vectors for two widely varying values of undressed energy given by $\varepsilon_1 = \omega cos\theta$. It is conjectured that similar behavior will also be present in the $\vec{u}(\psi_2, \chi_2)$ of qutrit Λ and V models.

3) Third-order invariant vector: $\vec{v} = \left\{\sqrt{\frac{1}{3}}, \quad \sqrt{\frac{1}{3}}, \quad \sqrt{\frac{1}{3}}\right\}$.

The third-order vector turns out to be a constant one, for the Ξ-model and angles are

$$\psi_3 = cos^{-1}\sqrt{\frac{1}{3}} \cong 55^0 \,, \tag{20a}$$

$$\chi_3 = tan^{-1}(1) = 45^0 \,. \tag{20b}$$

It is conjectured that the third invariant vector is always constant and has the angles given previously for pure qutrit state. The spherical angles associated with the other two vectors, $\vec{w}(\psi_1, \chi_1)$ and $\vec{u}(\psi_2, \chi_2)$, capture the time-dependent dynamic essence of this model.

5. Conclusion and Next Steps

The 3-D IVR vectors representing qutrit states captured the essential dynamics of the cascade or Ξ-model. The model qutrit state is pure, and so has fewer degrees of freedom. Out of three vectors, only one was found to display complex behavior. Out of the remaining two vectors, one is a constant and another has a linearly time-dependent azimuthal angle.

On the other hand, the IVR is capable of displaying the full static or dynamic behavior of a mixed qutrit state with all 8 degrees of freedom as well. In that situation, the qutrit dynamics are expressed by the behavior of three parameters each of $\vec{u}$ and $\vec{v}$ and two parameters of $\vec{w}$ as it has unit length by definition. This is a significant advance compared with the traditional approach based on Gell-Mann special unitary group of order 3 [SU(3)] matrices. The IVR will be applied in the future to study the relative performance of different qutrit models for quantum sensing.

6. References

1. Degen CL, Reinhard F, Cappellaro P. Quantum sensing. Rev Mod Phys. 2017;89(3).

2. Quezada LF, Nahmad-Achar E. Quantum phases of a three-level matter-radiation interaction model using $SU(3)$ coherent states with different cooperation numbers. Phys Rev A. 2018;97(6):063819. arXiv:1803.03129v1.

3. Bahder T. One qubit quantum sensor. Private notes. 2018 Nov.

4. Hu H, Zhang C. Spin-1 topological monopoles in parameter space of ultracold atoms. Phys Rev A. 2018;98(1):013627. arXiv:1802.08222v1.

5. Mishra VK. Three dimensional visualization of qutrit states. 2016 Nov. arXiv:1611.02701.

6. Kurzynski P. Multi-Bloch vector representation of the qutrit. Quantum Inf Comp. 2011;11:361. arXiv:0912.3155v1.

1 (PDF)	DEFENSE TECHNICAL INFORMATION CTR DTIC OCA
2 (PDF)	CCDC ARL IMAL HRA RECORDS MGMT FCDD RLD CL TECH LIB
1 (PDF)	GOVT PRINTG OFC A MALHOTRA
1 (PDF)	CCDC ARL FCDD RLC NC V MISHRA